Inverse Problems in Semiparametric Statistical Models

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Outline

Introduction
  Semiparametric Models
  Examples

Theoretical Foundations
  Intuition
  Rigorous Statement
  Semiparametric Efficient Estimation

Semiparametric Inferences
  Bootstrap Inferences
  Profile Sampler
  Sieve Estimation

Future (Theoretical) Directions
Semiparametric Models

- We observe i.i.d. data \( \{X_i\}_{i=1}^n \sim \{P_{\theta,\eta} : \theta \in \Theta \subset \mathbb{R}^k, \eta \in \mathcal{H} \} \)
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- Even we are only interested in \( \theta \), the estimation of \( \eta \) is usually unavoidable.
The hazard function of the survival time $T$ of a subject with covariate $Z$ is modelled as:

$$
\lambda(t|z) \equiv \lim_{\Delta \to 0} \frac{1}{\Delta} Pr(t \leq T < t + \Delta | T \geq t, Z = z) = \lambda(t) \exp(\theta'z),
$$

where $\lambda$ is an unspecified baseline hazard function.

Consider the current status data in which the event time $T$ is unobservable but we know whether the event has occurred at the examination time $C$ or not. Thus, we observe $X = (C, \delta, Z)$, where $\delta = I\{T \leq C\}$. 

Based on the above proportional hazard assumption, we can write down the log-likelihood as follows

$$
\log \text{lik}(\theta, \eta)(X) = \delta \log \left[ 1 - \exp(-\exp(\theta'Z)\eta(C)) \right] - (1 - \delta) \exp(\theta'Z)\eta(C),
$$

where the nuisance (monotone) function $\eta(y) \equiv \int_0^y \lambda(t)dt$, also called as cumulative hazard function.
Example II: Conditionally Normal Model

We assume that $Y|(W = w, Z = z) \sim N(\theta'w, \eta(z))$. The log-likelihood can be easily written as

$$\log \text{lik}(\theta, \eta)(X) = -\frac{1}{2} \log \eta(Z) - \frac{(Y - \theta'W)^2}{2\eta(Z)},$$

where $\eta(z)$ is positive.
Model III. Partly Linear Model

We assume that

\[ Y = \theta' W + \eta(Z) + \epsilon, \]

where \( \epsilon \) is independent of \((W, Z)\) and \( \eta \) is an unknown smooth function belonging to second order Sobolev space. We assume that \( \epsilon \) is normally distributed (can be relaxed to some tail conditions).
Model IV. Semiparametric Copula Model

We observe random vector \( X = (X_1, \ldots, X_d) \) with multivariate distribution function \( F(x_1, \ldots, x_d) \), and want to estimate the dependence structure in \( X \). To avoid the curse of dimensionality, we will apply the following Copula approach.

According to Sklar (1959), there exists a unique Copula function \( C_0(\cdot) \) such that

\[
F(x_1, \ldots, x_d) = C_0(F_1(x_1), \ldots, F_d(x_d)),
\]

where \( F_j(\cdot) \) is the marginal distribution for \( X_j \).
To model the dependence within $X$, we use the parametric Copula $C_\theta(\cdot)$, i.e., $C_{\theta_0} = C_0$. Thus, the log-likelihood is written as

$$\log \text{lik}(\theta, F_1, \ldots, F_d)(X) = \log c_\theta(F_1(X_1), \ldots, F_d(X_d)) + \sum_{j=1}^{d} \log f_j(X_j),$$

where $f_j$ is the marginal density function and

$$c_\theta(t_1, \ldots, t_d) = \frac{\partial^d}{\partial t_1 \cdots \partial t_d} C_\theta(t_1, \ldots, t_d).$$
We hope to obtain the \textit{semiparametric efficient estimate} $\hat{\theta}$, which achieves the minimal asymptotic variance bound in the sense that

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, V^*)$$

where $V^*$ is the minimal one over all the regular semiparametric estimators.
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IDEA: The minimal $V^*$ actually corresponds to the largest asymptotic variance over all the parametric submodels

$$\{ t \mapsto \log \text{lik}(t, \eta_t) : t \in \Theta \}$$

of the semiparametric model in consideration. The parametric submodel achieving $V^*$ is called as the least favorable submodel (LFS), see Bickel et al (1996).
Now, let us turn our attention to LFS defined as

\[ t \mapsto \log \text{lik}(t, \eta^*_t), \]

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The LFS needs to pass the true value \((\theta_0, \eta_0)\), i.e., \( \eta^*_{\theta_0} = \eta_0 \), and has the corresponding information matrix as

\[ \tilde{I}_0 = E\tilde{\ell}_0\tilde{\ell}_0', \]  

where \( \tilde{\ell}_0 \equiv \frac{\partial}{\partial t}|_{t=\theta_0} \log(t, \eta^*_t) \).

(This is just the usual way to calculate the information in parametric models)
Intuition 1

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Obviously, \( V^* = \tilde{I}_0^{-1} \).
What is the mysterious $\eta^*_t$?
Intuition II

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- In fact, Severini and Wong (1992) discovered that

$$
\eta^*_t = \arg \sup_{\eta \in \mathcal{H}} E \log \text{lik}(t, \eta) \quad \text{for any fixed } t \in \Theta
$$

after some simple derivations! This is not surprising since $\eta^*_t$ behaves like the true value for $\eta$ at each fixed $\theta$. 
According to our discussions on LFS in the above, we expect to obtain an efficient estimate of $\theta$ if we can estimate the abstract LFS, i.e., $\eta_t^*$, accurately.
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Let $S_n(\theta) \equiv \sum_{i=1}^{n} \log lik(\theta, \hat{\eta}_\theta)(X_i)$.

In fact, we can easily show that

$$\hat{\theta} \equiv \arg \max_{\theta \in \Theta} S_n(\theta),$$

is semiparametric efficient if $\hat{\eta}_t$ is a consistent estimate for $\eta^*_t$. 

Intuition III

- According to our discussions on LFS in the above, we expect to obtain an efficient estimate of $\theta$ if we can estimate the abstract LFS, i.e., $\eta_t^*$, accurately.
- Let $S_n(\theta) \equiv \sum_{i=1}^{n} \log \operatorname{lik}(\theta, \hat{\eta}\theta)(X_i)$.
- In fact, we can easily show that

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- Therefore, we can claim that the efficient estimation of $\theta$ boils down to the estimation of the least favorable curve $\eta_t^*$. 
Efficient estimation of $\theta$ in presence of an infinite dimensional $\eta$

\[ \downarrow \]
Least favorable submodel: $t \mapsto \log \text{lik}(t, \eta^*_t)$

\[ \downarrow \]
Consistent estimation of $\eta^*_t$
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$\Downarrow$

Least favorable submodel: $t \mapsto \log lik(t, \eta_t^*)$

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Consistent estimation of $\eta_t^*$

- The estimation accuracy of $\eta_t^*$, i.e., convergence rate, determines the second order efficiency of $\hat{\theta}$ (Cheng and Kosorok, 2008);
Summary

Efficient estimation of $\theta$ in presence of an infinite dimensional $\eta$

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Least favorable submodel: $t \mapsto \log lik(t, \eta^*_t)$

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Consistent estimation of $\eta^*_t$

- The estimation accuracy of $\eta^*_t$, i.e., convergence rate, determines the second order efficiency of $\hat{\theta}$ (Cheng and Kosorok, 2008);

- How we estimate $\eta^*_t$ depends on the parameter space $\mathcal{H}$, and different regularizations on $\eta^*_t$ gives different forms of $\hat{\theta}$, see four examples to be presented.
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In fact, the efficient score function can be understood as the residual of the projection of the score function for \( \theta \) onto the tangent space, which is defined as the closed linear span of the tangent set generated by the score function for \( \eta \).
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The LFS exists if the tangent set is closed. This is true for all of our examples in this talk.
As discussed above, we need to estimate $\eta_t^*$ consistently in order to obtain the efficient $\hat{\theta}$. Recall that

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\eta_t^* = \arg \max_{\eta \in H} E \log \text{lik}(t, \eta).
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Therefore, a natural estimate for $\eta_\theta^*$ is

$$
\hat{\eta}_\theta = \arg \max_{\eta \in H} \sum_{i=1}^{n} \log \text{lik}(\theta, \eta)(X_i)
$$

for any fixed $\theta \in \Theta$. 

In the above, \( \hat{\eta}_\theta \) is the NPMLE, \( S_n(\theta) \) is just the profile likelihood log \( pl_n(\theta) \), and \( \hat{\theta} \) becomes the semiparametric MLE.

The above maximum likelihood estimation works for our example I, i.e., Cox model, due to monotone constraints (see the work by Jon Wellner and his coauthors). However, the NPMLE is not always well defined. Thus, some form of regularization is needed especially when \( \eta \) needs to be estimated smoothly.
Kernel estimation: This is particularly useful when $\eta_{\theta}^*$ has an explicit form. In our example II, i.e., conditionally normal model, we have

$$\hat{\eta}_{\theta,b_n}(z) = \frac{\sum_{i=1}^{n}(Y - \theta' W)^2 K((z - Z_i)/b_n)}{\sum_{i=1}^{n} K((z - Z_i)/b_n)} > 0,$$

where $K(\cdot)$ is some kernel function and $b_n$ is the related bandwidth.
Penalized estimation: In our example III, i.e., partly linear model, we have

\[
\hat{\eta}_{\theta, \lambda_n} = \arg \max_{\eta \in \mathcal{H}} \left\{ \sum_{i=1}^{n} \log \text{lik}(\theta, \eta)(X_i) - \lambda_n \int_{\mathcal{Z}} [\eta^{(2)}(z)]^2 \, dz \right\}, \tag{3}
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In this example, $\hat{\theta}$ is just the partial smoothing spline estimate.
Sieve estimation: Here, we perform similar maximum likelihood estimation but replace the infinite dimensional parameter space $\mathcal{H}$ by its sieve approximation $\mathcal{H}_n$, e.g., B-spline space.
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In our example IV, i.e., semiparametric copula model, we have

$$\hat{\eta}_{\theta,s_n} = \arg\max_{\eta \in \mathcal{H}_n} \sum_{i=1}^{n} \log \text{lik}(\theta, \eta)(X_i),$$  \hspace{1cm} (4)

where $\mathcal{H}_n = \{\eta(\cdot) = \sum_{s=1}^{s_n} \gamma_s B_s(\cdot)\}$ is the B-spline space.
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An advantage of B-spline estimation is that we can transform the semiparametric estimation into the parametric estimation with increasing dimension as sample size.
Remark

- Under regularity conditions, all the above four estimation approaches yield the semiparametric efficient $\hat{\theta}$, see Cheng (2011) for more details.
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- Cheng and Kosorok (2008) show that the second order semiparametric efficiency of \( \hat{\theta} \) is determined by the smoothing parameters, i.e., \( b_n \), \( \lambda_n \) and \( s_n \), and the size of \( \mathcal{H} \) (in terms of entropy number).
Remark

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- Cheng and Kosorok (2008) show that the second order semiparametric efficiency of $\hat{\theta}$ is determined by the smoothing parameters, i.e., $b_n$, $\lambda_n$ and $s_n$, and the size of $\mathcal{H}$ (in terms of entropy number).
- In some situations, it might be more proper to use other criterion function than the likelihood function, e.g., use the least square criterion function in the partly linear model (replace $\epsilon \sim \mathcal{N}(0, \sigma^2)$ by the sub-exponential tail condition).
In the end, I describe three (almost) automatic semiparametric inferential tools for obtaining the semiparametric efficient estimate and constructing the confidence interval/credible set in the literature.
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- Profile Sampler [Lee, Kosorok and Fine (2005)]
- Sieve Estimation [Chen (2007)]
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- Small sample advantages;
- Replace the tedious theoretical derivations in semiparametric inferences with routine simulations of bootstrap samples, e.g., the bootstrap confidence interval.
The bootstrap estimator is defined as

\[ (\hat{\theta}^*, \hat{\eta}^*) = \arg \sup_{\theta \in \Theta, \eta \in \mathcal{H}} \sum_{i=1}^{n} \log \text{lik}(\theta, \eta)(X_i^*), \quad (5) \]

where \((X_1^*, \ldots, X_n^*)\) is the bootstrap sample.
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\]

where \((X^*_1, \ldots, X^*_n)\) is the bootstrap sample.

Recently, Cheng and Huang (2010) showed that (i) \(\hat{\theta}^*\) has the same asymptotic distribution as the semiparametric efficient \(\hat{\theta}\); (ii) the bootstrap confidence interval is theoretically valid, for a general class of exchangeably weighted bootstrap resampling schemes, e.g., Efron’s bootstrap and Bayesian bootstrap.
We assign some prior $\rho(\theta)$ on the profile likelihood $\log pl_n(\theta)$. MCMC is used for sampling from the posterior of the profile likelihood. This resulting MCMC chain is called as the profile sampler.

The inferences of $\theta$ are based on the profile sampler. Lee, Kosorok and Fine (2005) showed that chain mean (the inverse of chain variance) approximates the semiparametric efficient $\hat{\theta}(\tilde{I}_0)$, and the credible set for $\theta$ has the desired asymptotic coverage probability.
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Sieve Estimation

- Translate the semiparametric estimation into the parametric estimation with increasing dimension:

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(\hat{\theta}, \hat{\gamma}) = \arg \max_{\theta \in \Theta, \gamma \in \Gamma} \sum_{i=1}^{n} \log \text{lik}(\theta, \gamma' B)(X_i).
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\]

- An advantage of B-spline estimation is that we are able to give an explicit B-spline estimate for the asymptotic variance \(V^*\) as a byproduct of the establishment of semiparametric efficiency of \(\hat{\theta}\). Indeed, it is simply the observed information matrix if we treat the semiparametric model as a parametric one after the B-spline approximation, i.e., \(H = H_n\).
Future (Theoretical) Directions

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- Joint inferences for $(\theta, \eta)$ (extremely difficult.....);
Thanks for your attention....

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